A MULTIVARIATE DEGRADATION MODEL WITH DEPENDENCE DUE TO SHOCKS

SOPHIE MERCIER*

Université de Pau et des Pays de l'Adour
Laboratoire de Mathématiques et de leurs Applications (UMR CNRS 5142)
Bâtiment IPRA, Avenue de l'Université, BP 1155, F-64013 Pau, FRANCE

ANNE BARROS

Norwegian University of Science and Technology
Department of Production and Quality Engineering, RAMS Group
*S. P. Andersens veg 5, Valgrinda*2.308A, 7491 Trondheim, NORWAY*

ANTOINE GRALL

Université de Technologie de Troyes
Institut Charles Delaunay (UMR CNRS 6281)
12, rue Marie Curie, F-10004 Troyes, FRANCE

The aim of this paper is to model degradation phenomena in a multi-unit context, with dependence between components induced by a common and external stressing random environment. The stress arrives by shocks and each shock induces a sudden increase to the deterioration level of each component. The model induces a two-step dependence between components: firstly, all components are simultaneously impacted by a shock; secondly, for a given shock, the deterioration increments of the different components are correlated. This leads to a multivariate degradation model. A method of moments is proposed, for the estimation of the model parameters. Assuming the components to belong to a system with a given failure zone, the system reliability is next provided and the influence of the model parameters on the system lifetime is studied. Numerical experiments illustrate the study.

Keywords: Multivariate compound Poisson process, Multivariate Lévy process, Failure zone, Upper set, Stochastic comparison

1. Introduction

The aim of this paper is to model multivariate degradation phenomena in a multi-unit context, which takes into account some stochastic dependence

*Corresponding Author; Tel.: +33 (0)5 59 40 75 37; Email: sophie.mercier@univ-pau.fr

between units. This kind of model is not so widely studied in the framework of reliability theory since the associated mathematical developments lead to cumbersome numerical computations, see however [1], [2], [5], [7] for a few exceptions. In the present work, the dependence between components is induced by a common and external stressing random environment which arrives by shocks. These shocks may be due to some external specific demand, some significative change of the operational condition, of the environments, etc... Each shock induces a sudden increase to the deterioration levels of the components. The model induces a two-step dependence between components: firstly, all components are simultaneously impacted by a shock; secondly, for a given shock, the deterioration increments of the different components can be correlated. Out of shocks, the components behave independently. From a practical point of view, we have a multivariate degradation phenomena whose speed is not changed by the shocks, but whose level can be significantly increased at each shock event.

The intrinsic deterioration of the units is modeled through independent stochastic processes, which are classically assumed to be Gamma processes [8]. The shocks arrive independently, according to a Poisson process. Sudden change in the deterioration level of a component at shocks times will be seen to be just equivalent to some sudden change in its age, leading to some kind of virtual age model [4]. Also, the choice of an adequate distribution for the increments of deterioration/age at shocks times may give rise to possibly fatal shocks, leading to some shock model with mixed effect.

The paper is organized as follows: in Section 2, the model is presented and the Laplace transform of the multivariate degradation process is given. In Section 3, an estimation scheme based on a method of moments is proposed. Assuming the components to be the constitutive part of a system with a given failure zone, the system reliability is next provided in Section 4 and the influence of the model parameters on the system lifetime is studied. Conclusive remarks end the paper in Section 5.

2. Degradation modeling

Let $\mathbf{Z}_t = \left(Z_t^{(1)}, \cdots, Z_t^{(n)}\right)_{t\geq 0}$, where $\left(Z_t^{(i)}\right)_{t\geq 0}$ are independent and identically distributed (i.i.d.) univariate Gamma processes. With no loss of generality, the random variable $Z_t^{(i)}$ is supposed to be gamma distributed with shape t and scale 1 (denoted by $\Gamma(t,1)$), for all $t \geq 0$ and all $i \in \{1, \cdots, n\}$. The intrinsic deterioration of component i is modelled by $\left(Z_{a_i t}^{(i)}\right)_{t\geq 0}$, where $Z_{a_i t}^{(i)}$ is gamma distributed $\Gamma(a_i t, 1)$. Setting $\mathbf{a} = (a_1, \cdots, a_n)$, the multivariate intrinsic deterioration at time t can be restated as $\mathbf{Z}_{t\mathbf{a}} = \left(Z_{a_1 t}^{(1)}, \cdots, Z_{a_2 t}^{(n)}\right)$.

The shocks are assumed to arrive according to a homogeneous Poisson process $N = (N_t)_{t\geq 0}$ with rate $\lambda > 0$, independent of $\mathbf{Z} = (\mathbf{Z}_t)_{t\geq 0}$.

We next introduce i.i.d. non negative random vectors $\mathbf{U}_1$, ..., $\mathbf{U}_j$, ... independent of N and of $\mathbf{Z}$, with $\mathbf{U}_j = \left(U_j^{(1)}, \cdots, U_j^{(n)}\right)$, all $j \in \mathbb{N}^*$. When unnecessary, we drop subscript j, and we set $\mathbf{U} = \left(U^{(1)}, \cdots, U^{(n)}\right)$ to be a generic copy of $\mathbf{U}_j$. Setting

$$Y_t^{(i)} = \sum_{j=1}^{N_t} U_j^{(i)}, \text{ for all } t \geq 0 \text{ and all } 1 \leq i \leq n,$$

the random variable $Y_t^{(i)}/a_i$ stands for the cumulated increment of age of component i on $[0, t]$. This means that the deterioration level of component i at calendar time t is the same as if its age were $t + Y_t^{(i)}/a_i$, which hence appears as its virtual age. The corresponding deterioration level is $Z^{(i)}_{a_i\left(t+Y_t^{(i)}/a_i\right)} = Z^{(i)}_{a_i t + Y_t^{(i)}}$.

In the multivariate framework, we now set:

$$\mathbf{Y}_t = \sum_{j=1}^{N_t} \mathbf{U}_j = \left(Y_t^{(1)}, \cdots, Y_t^{(n)}\right),$$

$$\mathbf{X}_t = \mathbf{Z}_{t\mathbf{a}+\mathbf{Y}_t} = \left(Z^{(1)}_{a_1 t + Y_t^{(1)}}, \cdots, Z^{(n)}_{a_n t + Y_t^{(n)}}\right). \tag{1}$$

It is easy to check that $\mathbf{X}_t$ is identically distributed $\left(\cdots \overset{\mathcal{D}}{=} \cdots\right)$ as the sum of two independent random vectors: $\mathbf{X}_t \overset{\mathcal{D}}{=} \mathbf{X}_t^{\perp} + \mathbf{X}_t^{\parallel}$, where $\mathbf{X}_t^{\perp} \overset{\mathcal{D}}{=} \mathbf{Z}_{t\mathbf{a}}$ corresponds to the intrinsic deterioration and $\mathbf{X}_t^{\parallel} \overset{\mathcal{D}}{=} \mathbf{Z}_{\mathbf{Y}_t}$ to the deterioration due to the shocks. The whole dependence between the components of $\mathbf{X}$ is included in the dependence between the components of $\mathbf{X}^{\parallel}$. Besides, introducing $\mathbf{V}_j \overset{\mathcal{D}}{=} \mathbf{Z}_{\mathbf{U}_j}$, $j \in \mathbb{N}^*$ as i.i.d. random vectors independent of N, we have

$$\mathbf{X}_t^{\parallel} \overset{\mathcal{D}}{=} \mathbf{Z}_{\sum_{j=1}^{N_t} \mathbf{U}_j} \overset{\mathcal{D}}{=} \sum_{j=1}^{N_t} \mathbf{V}_j = \sum_{j=1}^{N_t} \left(V_j^{(1)}, \cdots, V_j^{(n)}\right),$$

where $V_j^{(i)}$ is conditionnaly gamma distributed $\Gamma\left(U_j^{(i)}, 1\right)$ given $U_j^{(i)}$, for all $j \in \mathbb{N}^*$ and all $1 \leq i \leq n$. The process $\mathbf{X}^{\parallel}$ can hence be seen as a multivariate compound Poisson process. With this interpretation, $\mathbf{V}_j$ stands for the multivariate increment of deterioration at the j-th shock event.

Note that no assumption is put on the distribution of the $\mathbf{U}_j$'s. Taking $\mathbf{U}_j$ as the result of a Bernoulli trial between an immediate failure state for all (or only a few) components and an absolutely continuous random vector, then the model can be seen as a shock model with mixed effect, in the sense that a single shock can either entail the simultaneous failures of all (or only a few) components, or can either increase their deterioration.

Finally, remembering that a gamma process is a Lévy process, we can notice that $\mathbf{Z}$ is the conjonction of n independent Lévy processes, and hence, $\mathbf{Z}$ is a multivariate Lévy process. Also, $\mathbf{Y}$ is a non negative multivariate compound Poisson process and consequently, it is a non decreasing Lévy process, also called multivariate subordinator. Based on (1), the process $\mathbf{X}$ is obtained through multivariate subordination of the Lévy process $\mathbf{Z}$ and hence, $\mathbf{X}$ is a (multivariate) Lévy process. As a consequence, the full distribution of $\mathbf{X}$ is entirely characterized by the (multivariate) Laplace transform of $\mathbf{X}_t$, which we next provide.

Proposition 2.1. *For $\mathbf{s} = (s_1, \cdots, s_n) \in \mathbb{R}_+^n$, we have*

$$\mathcal{L}_{\mathbf{X}_t}(\mathbf{s}) = \mathbb{E}\left(e^{-\sum_{i=1}^n s_i X_t^{(i)}}\right) = e^{-\lambda t\ \left(1-\mathcal{L}_{\mathbf{U}_1}(\ln(1+\mathbf{s}))\right)} \prod_{i=1}^n (1+s_i)^{-a_i t}$$

where $\ln(1+\mathbf{s}) = (\ln(1+s_1), \cdots, \ln(1+s_n))$ *and*

$$\mathcal{L}_{\mathbf{U}_1}(\ln(1+\mathbf{s})) = \mathbb{E}\left(\prod_{i=1}^n (1+s_i)^{-U_1^{(i)}}\right).$$

Based on this full form expression for the Laplace transform of $\mathbf{X}_t$, one can show that the process is theoretically identifiable.

3. Estimation

We here derive various moments of $\mathbf{X}_t$ from its Laplace transform.

Proposition 3.1. *For all $1 \le i, j \le n$ with $i \ne j$ and all $t \ge 0$, we have:*

$$\mathbb{E}\left(X_t^{(i)}\right) = t\ m^{(i)} \text{ with } m^{(i)} = a_i + \lambda\ \mathbb{E}\left(U^{(i)}\right),$$

$$\operatorname{var}\left(X_t^{(i)}\right) = t\ \mu_2^{(i)} \text{ with } \mu_2^{(i)} = m^{(i)} + \lambda\ \mathbb{E}\left(U^{(i)2}\right),$$

$$\operatorname{cov}\left(X_t^{(i)}, X_t^{(j)}\right) = t\ c_{i,j} \text{ with } c_{i,j} = \lambda\ \mathbb{E}\left(U^{(i)}U^{(j)}\right),$$

$$\mathbb{E}\left(\left(X_t^{(i)} - t\ m^{(i)}\right)^3\right) = t\,\mu_3^{(i)}\ \ with\ \mu_3^{(i)} = 2a_i + \lambda\,\mathbb{E}\left(U^{(i)}\left(U^{(i)}+1\right)\left(U^{(i)}+2\right)\right).$$

In case of a parametric model, these full form expressions allow to developp estimation methods based on moments. When observations are periodic, unbiaised estimators of $m^{(i)}$, $\mu_2^{(i)}$ and $c_{i,j}$ are used, as provided in [3] or [6]. In the case where an additional equation is required, a similar unbiaised estimator of the third-order moment $\mu_3^{(j)}$ is developped. (Higher moments can also be used, if necessary).

Example 3.1. Starting from three independant and exponentially distributed random variables $V^{(i)}$, $i = 1, 2, 3$ with respective mean $1/\lambda_i$, $i = 1, 2, 3$, we set $U^{(1)} = V^{(1)} + V^{(3)}$, $U^{(2)} = V^{(2)} + V^{(3)}$. The first line of the following table provides the value of the parameters, the second and third lines provide the mean and the standard deviation of 500 estimation results based on 500 simulations of 1000 paths observed at times 1, 2, 3, ..., 100, respectively, and the last line gives the associated 95% confidence intervals (CI), based on empirical quantiles.

	a_1	a_2	λ	λ_1	λ_2	λ_3
param.	1	2	5	2	1	3
mean	0.97749	1.9687	5.0558	2.0102	1.0057	3.0189
std	0.12285	0.16467	0.27253	0.058268	0.028479	0.1352
95% CI	[0.967,0.988]	[1.954,1.983]	[5.032,5.080]	[2.005,2.015]	[1.003,1.008]	[3.007,3.031]

4. Reliability assessment and impact of the parameters on the reliability

The n components are now assumed to be the constitutive parts of a system and we set $\mathcal{D} \subset \mathbb{R}_+^n$ be the set of the down states for the system. A common assumption is that, without repair, a down system cannot recover. This writes: $\mathbf{X}_t \in \mathcal{D}$ implies $\mathbf{X}_{t+s} \in \mathcal{D}$, for all $s, t \geq 0$. As $\mathbf{X}_t \leq \mathbf{X}_{t+s}$ (componentwise), a natural assumption on $\mathcal{D}$ is that: if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \in \mathcal{D}$, then $\mathbf{y} \in \mathcal{D}$, or equivalently, that the failure area $\mathcal{D}$ must be an upper set. In the two-dimensional case, an example of such an upper set is

$$\mathcal{D} = \left\{\mathbf{x} = (x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 \geq L\right\}, \tag{2}$$

see e.g. [7]. If the system is coherent with structure function Φ and if each component i has its own failure threshold L_i (independent of the failure thresholds of the other components), then

$$\mathcal{D}=\left\{\mathbf{x}\in\mathbb{R}_{+}^{n}:\Phi\left(\mathbf{I}\left(\mathbf{x}\right)\right)=0\right\},$$

where $\mathbf{I}\left(\mathbf{x}\right)=\left(\mathbf{1}_{[0,L_i[}\left(x_i\right)\right)_{1\leq i\leq n}\in\{0,1\}^n$.

Proposition 4.1. *The reliability of the system at time t is:*

$$R\left(t\right)=\mathbb{P}\left(X_t\notin\mathcal{D}\right)=\mathbb{E}\left[\int_{\mathbb{R}_{+}\setminus\mathcal{D}}\left(\prod_{i=1}^{n}f^{(\Gamma)}_{a_i\left(t+Y_t^{(i)}\right)}\left(z_i\right)\right)dz\right],\qquad(3)$$

where $f_{\theta}^{(\Gamma)}$ is the Gamma probability distribution function with shape θ and scale 1.

Apart from particular cases, the distribution of the multivariate compound Poisson process $\mathbf{Y}_t=\left(Y_t^{(1)},\cdots,Y_t^{(n)}\right)$ is not tractable analytically. The previous formula cannot hence be made more explicite. It can be numerically assessed through Monte-Carlo simulations of $\mathbf{Y}_t$ (which are much simpler than direct Monte-Carlo simulations of $\mathbf{X}_t$).

We now provide results showing the influence of the model parameters on the system lifetime, or on its reliability, equivalently. With that aim, two systems S and $\tilde{S}$ are considered with identical failure zones and identical parameters, except from one. We add $\tilde{}$ to every quantity refering to $\tilde{S}$.

Proposition 4.2. *Assume that $\lambda\leq\tilde{\lambda}$. Then, $R\left(t\right)\geq\tilde{R}\left(t\right)$ for any $t\geq0$.*

This proposition means that the lifetime of the system stochastically decreases when the shock frequency increases.

Proposition 4.3. *Assume that each component has its own failure threshold and that the system is coherent. With the previous notations:*

(1) In case of a series system, if $F_{\mathbf{U}}\leq F_{\widetilde{\mathbf{U}}}$, then $R\left(t\right)\leq\tilde{R}\left(t\right)$,
(2) In case of a parallel system, if $\bar{F}_{\mathbf{U}}\leq\bar{F}_{\widetilde{\mathbf{U}}}$, then $R\left(t\right)\geq\tilde{R}\left(t\right)$, all $t\geq0$.

In case $\mathbf{U}$ and $\widetilde{\mathbf{U}}$ are identically marginaly distributed, each condition $F_{\mathbf{U}}\leq F_{\widetilde{\mathbf{U}}}$ and $\bar{F}_{\mathbf{U}}\leq\bar{F}_{\widetilde{\mathbf{U}}}$ means that the dependence between the margins of $\widetilde{\mathbf{U}}$ is stronger than between the margins of $\mathbf{U}$. In that case and if both conditions $F_{\mathbf{U}}\leq F_{\widetilde{\mathbf{U}}}$ and $\bar{F}_{\mathbf{U}}\leq\bar{F}_{\widetilde{\mathbf{U}}}$ are fullfilled, the reliability increases with the dependence for a series system and decreases with the dependence for a parallel system.

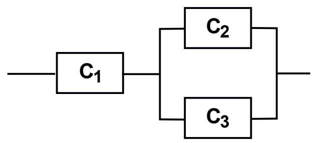

Figure 1. Structure of the three-unit system.

We now illustrate the previous results on a three unit coherent system, according to the structure depicted in Figure 1. Each unit has its own specific failure threshold L_i. This provides

$$\mathcal{D} = \left([L_1, \infty[\times \mathbb{R}_+^2\right) \cup \{\mathbb{R}_+ \times [L_2, \infty[\times [L_3, \infty[\}$$

and, based on (3), we get:

$$R\left(t\right) = \mathbb{E}\left[F^{(\Gamma)}_{a_1 t + Y_t^{(1)}}\left(L_1\right)\left(1 - \bar{F}^{(\Gamma)}_{a_2 t + Y_t^{(2)}}\left(L_2\right)\bar{F}^{(\Gamma)}_{a_3 t + Y_t^{(3)}}\left(L_3\right)\right)\right],$$

where $F^{(\Gamma)}_{\theta}$ $\left(\bar{F}^{(\Gamma)}_{\theta}\right)$ is the Gamma cumulative distribution (survival) function with shape θ and scale 1.

We take $a_i = 0$ and each $U^{(i)}$ is exponentially distributed with mean 1 for all $i \in \{1, 2, 3\}$. We consider two cases: In the first case $U^{(1)}$ is independent of $\left(U^{(2)}, U^{(3)}\right)$. The dependence between $U^{(2)}$ and $U^{(3)}$ is modelled through a Marshall-Olkin distribution, with

$$\bar{F}_{\left(U^{(2)}, U^{(3)}\right)}\left(u_2, u_3\right) = e^{-(1-\lambda_{12})u_2 - (1-\lambda_{12})u_3 - \lambda_{12}\max(u_2, u_3)}. \tag{4}$$

This provides exponential marginal distributions with mean 1 for any $\lambda_{12} \in [0, 1]$. Also, all the dependence is measured by λ_{12} and both $F_{\mathbf{U}}$ and $\bar{F}_{\mathbf{U}}$ increases when λ_{12} increases. We also take $(L_1, L_2, L_3) = (500, 100, 100)$. In the second case, $U^{(2)}$ is independent of $\left(U^{(1)}, U^{(3)}\right)$ and $\left(U^{(1)}, U^{(3)}\right)$ is Marshall-Olkin distributed (4). Also, $(L_1, L_2, L_3) = (100, 20, 100)$. The reliability $R\left(t\right)$ at time $t = 10$ is plotted with respect of the dependence (measured by λ_{12}) for both cases in Figure 2. The left plot stands for the first case, where component 1 is highly more reliable than the other two ones. The system hence roughly behaves like a parallel system and $R\left(t\right)$ decreases when the dependence increases. The right plot stands for the second case, where component 2 is highly less reliable than the other two ones. The system roughly behaves like a series system and $R\left(t\right)$ increases when the dependence increases. This example shows that in case of a general structure (and hence of a general failure zone), nothing can be said about the influence of the dependence on the reliability of the system.

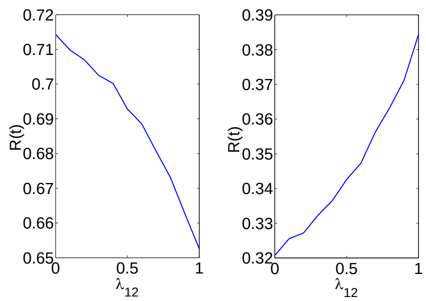

Figure 2. $R(t)$ with respect of λ_{12} for $t = 10$

5. Conclusion

The proposed model allows to consider stochastic dependencies in a multi-unit context with tractable calculation for the reliability. Further work is necessary to investigate some different topics such as alternate estimation procedures and the use of the above results in a maintenance context. Also, more precise applications with suitable form of dependencies would be welcomed, to enhance the interest of the proposed model.

References

[1] C.T. Barker and M.J. Newby. Optimal non-periodic inspection for a multivariate degradation model. *Reliab. Eng. Syst. Saf.*, 94(1):33–43, 2009.

[2] F. A. Buijs, J. W. Hall, J. M. van Noortwijk, and P. B. Sayers. Time dependent reliability analysis of flood defences using gamma processes. In G. Augusti, G. I. Schüeller, and M. Ciampoli, Editors, *Safety and Reliability of Engineering Systems and Structures*, 2209–2216, Rotterdam, 2005. Millpress.

[3] E. Çinlar, Z. P. Bažant, and E. Osman. Stochastic process for extrapolating concrete creep. *J. of the Engrg. Mech. Div., ASCE*, 103:1069–1088, 1977.

[4] L. Doyen and O. Gaudoin. Classes of imperfect repair models based on reduction of failure intensity or virtual age. *Reliab. Eng. Syst. Saf.*, 84(1):45–56, 2004.

[5] N. Ebrahimi. Indirect assessment of the bivariate survival function. *Ann. Inst. Statist. Math.*, 56(3):435–448, 2004.

[6] S. Mercier, C. Meier-Hirmer, and M. Roussignol. Bivariate gamma wear processes for track geometry modelling, with application to intervention scheduling. *Struct. Infrastruct. E.*, 8(4):357–366, 2012.

[7] S. Mercier and H. H. Pham. A preventive maintenance policy for a continuously monitored system with correlated wear indicators. *European J. Oper. Res.*, 222(2):263–272, 2012.

[8] J.M. van Noortwijk. A survey of the application of Gamma processes in maintenance. *Reliab. Eng. Syst. Saf.*, 94(1):2–21, 2009.